

SOME RESULTS ON CHERN'S PROBLEM

QI DING AND Y. L. XIN

ABSTRACT. For a compact minimal hypersurface M in S^{n+1} with the squared length of the second fundamental form S we confirm that there exists a positive constant $\delta(n)$ depending only on n , such that if $n \leq S \leq n + \delta(n)$, then $S \equiv n$, i.e., M is a Clifford minimal hypersurface, in particular, when $n \geq 6$, the pinching constant $\delta(n) = \frac{n}{23}$.

1. INTRODUCTION

Let M be a compact (without boundary) minimal hypersurface in the unit sphere S^{n+1} with the second fundamental form B , which can be viewed as a cross-section of the vector bundle $\text{Hom}(\odot^2 TM, NM)$ over M , where TM and NM denote the tangent bundle and the normal bundle along M , respectively. Minimal submanifolds in the sphere are interesting not only in its own right, but also are related to other interesting problems (see [10], for example). The simplest hypersurface in S^{n+1} is the n -equator, totally geodesic hypersurface. The important examples of minimal hypersurfaces in S^{n+1} are Clifford minimal hypersurfaces

$$S^k \left(\sqrt{\frac{k}{n}} \right) \times S^{n-k} \left(\sqrt{\frac{n-k}{n}} \right), \quad k = 1, 2, \dots, n-1.$$

J. Simons [7] discovered the intrinsic rigidity result. Shortly afterwards Chern-do Carmo-Kobayashi [3] and Lawson [4] independently showed that Simons' result is sharp and the equality is realized by the Clifford minimal hypersurfaces in the unit sphere. In their same paper [3], Chern-do Carmo-Kobayashi proposed to study subsequent gaps for the scalar curvature (or the squared length of the second fundamental form $S = |B|^2$). The problem was also collected by S. T. Yau in the well-known problem section in [11] and [12].

Peng-Terng [5] made the first effort to attack the Chern's problem and confirmed the second gap. Precisely, they proved that if the scalar curvature of M is a constant, then there exists a positive constant $C(n)$ depending only on n such that if $n \leq S \leq n + C(n)$, then $S = n$, where S stands for squared norm of the second fundamental

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form. Later, the pinching constant $C(n)$ was improved to $\frac{1}{3}n$, $n > 3$ by Cheng-Yang [2], and to $\frac{3}{7}n$, $n > 3$ by Suh-Yang [8], respectively.

More generally, Peng-Terng [6] obtained pinching results for minimal hypersurfaces without the constant scalar curvature assumption. They obtained that if M is a compact minimal hypersurface in S^{n+1} , then there exists a positive constant $\delta(n)$ depending only on n such that if $n \leq S \leq n + \delta(n)$, $n \leq 5$, then $S \equiv n$ which characterize the Clifford minimal hypersurfaces. Later, Cheng-Ishikawa [1] improved the the previous pinching constant when $n \leq 5$, and Wei-Xu [9] extended the result to $n = 6, 7$. Recently, Zhang [13] extended the results to $n \leq 8$ and improved the previous pinching constant. The key point is to estimate the upper bound of $\mathcal{A} - 2\mathcal{B}$ in terms of S and $|\nabla B|$ in all the above mentioned papers (please see the definition of \mathcal{A}, \mathcal{B} in next section).

In this paper, we continue to study the second gap problem without the constancy of the scalar curvature. We obtain new estimates for $\mathcal{A} - 2\mathcal{B}$ in terms S , $|\nabla B|$ and another higher order invariant of the second fundamental form of the minimal hypersurface M in S^{n+1} . Then, by the integral formulas established in [6] we can carried out more delicate integral estimates, which enable us to confirm the second gap in all dimensions. Firstly, we give the quantitative result to show our technique fits all dimensions. Then, we refine the estimates to obtain concrete pinching constant for dimension $n \geq 6$ where they are better than all previous results for dimension $n \geq 7$.

Theorem 1.1. *Let M be a compact minimal hypersurface in S^{n+1} with the squared length of the second fundamental form S . Then there exists a positive constant $\delta(n)$ depending only on n , such that if $n \leq S \leq n + \delta(n)$, then $S \equiv n$, i.e., M is a Clifford minimal hypersurface.*

Theorem 1.2. *If the dimension is $n \geq 6$, then the pinching constant $\delta(n) = \frac{n}{23}$.*

2. PRELIMINARIES

Let M be a minimal hypersurface in S^{n+1} with the second fundamental form B . We choose a local orthonormal frame field $\{e_1, \dots, e_n, \nu\}$ of S^{n+1} along M , such that e_i are tangent to M and ν is normal to M .

Set $B_{e_i e_j} = h_{ij}\nu$. Then the coefficients of the second fundamental form h_{ij} is a symmetric 2-tensor on M . Its trace vanishes everywhere by the minimal assumption on the submanifold M . Let S denote the squared length of the second fundamental form of M

$$S = |B|^2 = \sum_{i,j} h_{ij}^2.$$

By the Gauss equation it is an intrinsic invariant related to the scalar curvature of M .

J. Simons obtained the following Bochner type formula [7]

$$(2.1) \quad \frac{1}{2}\Delta S = |\nabla B|^2 + S(n - S),$$

where

$$|\nabla B|^2 = \sum_{i,j,k} h_{ijk}^2,$$

h_{ijk} is symmetric in i, j and k by the Codazzi equations. To study the second gap problem Peng-Terng [6] computed the second Bochner type formula as follows

$$(2.2) \quad \frac{1}{2}\Delta |\nabla B|^2 = |\nabla^2 B|^2 + (2n + 3 - S)|\nabla B|^2 + 3(2\mathcal{B} - \mathcal{A}) - \frac{3}{2}|\nabla S|^2,$$

where

$$|\nabla^2 B|^2 = \sum_{i,j,k,l} h_{ijkl}^2, \quad \mathcal{A} = \sum_{i,j,k,l,m} h_{ijk}h_{ijl}h_{km}h_{ml}, \quad \mathcal{B} = \sum_{i,j,k,l,m} h_{ijk}h_{klm}h_{im}h_{jl}.$$

It follows that

$$(2.3) \quad \int_M \sum_{i,j,k,l} h_{ijkl}^2 = \int_M \left((S - 2n - 3)|\nabla B|^2 + 3(\mathcal{A} - 2\mathcal{B}) + \frac{3}{2}|\nabla S|^2 \right).$$

For any fixed point $x \in M$, we take orthonormal frame field near x , such that $h_{ij} = \lambda_i \delta_{ij}$ at x for all i, j . Then

$$\sum_i \lambda_i = 0, \quad \sum_i \lambda_i^2 = S$$

and

$$\mathcal{A} = \sum_{i,j,k} h_{ijk}^2 \lambda_i^2, \quad \mathcal{B} = \sum_{i,j,k} h_{ijk}^2 \lambda_i \lambda_j.$$

There are pointwise estimates

$$(2.4) \quad \begin{aligned} \lambda_j^2 - 4\lambda_i \lambda_j &\leq \alpha S, \quad \forall i, j \\ 3(\mathcal{A} - 2\mathcal{B}) &\leq \alpha S |\nabla B|^2 \end{aligned}$$

with $\alpha = \frac{\sqrt{17}+1}{2}$ in [6] and

$$(2.5) \quad \begin{aligned} \sum_{i,j,k,l} h_{ijkl}^2 &\geq \frac{3}{4} \sum_{i,j} (\lambda_i - \lambda_j)^2 (1 + \lambda_i \lambda_j)^2 + \frac{3S(S-n)^2}{2(n+4)} \\ &= \frac{3}{2} (Sf_4 - f_3^2 - S^2 - S(S-n)) + \frac{3S(S-n)^2}{2(n+4)} \end{aligned}$$

in [1]. There is an integral equality

$$(2.6) \quad \int_M (\mathcal{A} - 2\mathcal{B}) = \int_M (Sf_4 - f_3^2 - S^2 - \frac{1}{4}|\nabla S|^2),$$

where $f_3 = \sum_i \lambda_i^3$, $f_4 = \sum_i \lambda_i^4$ in [6]. In a general local orthonormal frame field $f_3 = \sum_{i,j,k} h_{ij} h_{jk} h_{ki}$ and $f_4 = \sum_{i,j,k,l} h_{ij} h_{jk} h_{kl} h_{li}$.

3. NEW ESTIMATES OF $\mathcal{A} - 2\mathcal{B}$

In this paper we always assume $S \geq n$.

Define

$$\mathcal{F} = \sum_{i,j} (\lambda_i - \lambda_j)^2 (1 + \lambda_i \lambda_j)^2,$$

then

$$\mathcal{F} = 2(Sf_4 - f_3^2 - S^2 - S(S - n)).$$

It is a higher order invariant of the second fundamental form.

Lemma 3.1. *When the dimension $n \geq 4$,*

$$3(\mathcal{A} - 2\mathcal{B}) \leq 2S|\nabla B|^2 + C_1(n)|\nabla B|^2 \mathcal{F}^{\frac{1}{3}},$$

where $C_1(n) = (\sqrt{17} - 3)(6(\sqrt{17} + 1))^{-\frac{1}{3}}(\frac{2}{\sqrt{17}} - \frac{\sqrt{2}}{17} - \frac{1}{n})^{-\frac{2}{3}}$.

Proof. If there exist $i \neq j$ such that $\lambda_j^2 - 4\lambda_i \lambda_j = tS > 2S$, then by

$$S \geq \lambda_i^2 + \lambda_j^2 = \left(\frac{tS - \lambda_j^2}{4\lambda_j} \right)^2 + \lambda_j^2,$$

we have

$$\lambda_j^2 \leq \frac{1}{17}(t + 8 + 4\sqrt{4 + t - t^2})S,$$

moreover,

$$(3.1) \quad -\lambda_i \lambda_j \geq \frac{1}{4} \left(t - \frac{1}{17}(t + 8 + 4\sqrt{4 + t - t^2}) \right) S = \frac{1}{17}(4t - 2 - \sqrt{4 + t - t^2})S.$$

On the other hand,

$$(3.2) \quad (\lambda_i - \lambda_j)^2 = \frac{3}{4}\lambda_j^2 - 2\lambda_i \lambda_j + \lambda_i^2 + \frac{1}{4}\lambda_j^2 \geq \frac{3}{4}\lambda_j^2 - 3\lambda_i \lambda_j = \frac{3t}{4}S.$$

By the assumptions $n \geq 4$ and $S \geq n$, and (3.1) implies $-\lambda_i \lambda_j \geq 0.26S$, then combining (3.1) and (3.2), we obtain

$$\begin{aligned} \mathcal{F} &= \sum_{k,l} (\lambda_k - \lambda_l)^2 (1 + \lambda_k \lambda_l)^2 \\ (3.3) \quad &\geq 2(\lambda_i - \lambda_j)^2 (1 + \lambda_i \lambda_j)^2 \geq \frac{3t}{2}S(1 + \lambda_i \lambda_j)^2 \\ &\geq \frac{3t}{2} \left(-\lambda_i \lambda_j - \frac{S}{n} \right)^2 S \geq \frac{3t}{2} \left(\frac{1}{17}(4t - 2 - \sqrt{4 + t - t^2}) - \frac{1}{n} \right)^2 S^3. \end{aligned}$$

Define a function

$$(3.4) \quad \zeta(t) \triangleq \frac{t}{(t-2)^3} \left(\frac{1}{17}(4t-2-\sqrt{4+t-t^2}) - \frac{1}{n} \right)^2$$

on the interval $(2, \frac{\sqrt{17}+1}{2}]$. Then we have following rough estimate,

$$(3.5) \quad \begin{aligned} \min_{(2, \frac{\sqrt{17}+1}{2}]} \zeta(t) &\geq \min_{(2, \frac{\sqrt{17}+1}{2}]} \frac{t}{(t-2)^3} \left(\frac{1}{17}(4t-2-\sqrt{2}) - \frac{1}{n} \right)^2 \\ &= 4 \frac{\sqrt{17}+1}{(\sqrt{17}-3)^3} \left(\frac{2}{\sqrt{17}} - \frac{\sqrt{2}}{17} - \frac{1}{n} \right)^2. \end{aligned}$$

From (3.3), (3.4) and (3.5) we obtain

$$(3.6) \quad \begin{aligned} (\lambda_j^2 - 4\lambda_i\lambda_j - 2S)^3 &= (t-2)^3 S^3 \leq \frac{2\mathcal{F}}{3\zeta(t)} \\ &\leq \frac{(\sqrt{17}-3)^3}{6(\sqrt{17}+1)} \left(\frac{2}{\sqrt{17}} - \frac{\sqrt{2}}{17} - \frac{1}{n} \right)^{-2} \mathcal{F} \triangleq (C_1(n)\mathcal{F}^{1/3})^3, \end{aligned}$$

where $C_1(n) = (\sqrt{17}-3)(6(\sqrt{17}+1))^{-\frac{1}{3}}(\frac{2}{\sqrt{17}} - \frac{\sqrt{2}}{17} - \frac{1}{n})^{-\frac{2}{3}}$. By the definition of \mathcal{A} and \mathcal{B} and (3.6), we have

$$(3.7) \quad \begin{aligned} 3(\mathcal{A} - 2\mathcal{B}) &= \sum_{i,j,k} h_{ijk}^2 (\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_j\lambda_k - 2\lambda_i\lambda_k) \\ &\leq \sum_{i,j,k \text{ distinct}} h_{ijk}^2 (2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2) - (\lambda_i + \lambda_j + \lambda_k)^2) \\ &\quad + 3 \sum_{j, i \neq j} h_{ijj}^2 (\lambda_j^2 - 4\lambda_i\lambda_j) \\ &\leq 2S \sum_{i,j,k \text{ distinct}} h_{ijk}^2 + 3 \sum_{i \neq j} h_{ijj}^2 (2S + C_1(n)\mathcal{F}^{1/3}) \\ &\leq 2S|\nabla B|^2 + C_1(n)\mathcal{F}^{1/3}. \end{aligned}$$

The lemma holds obviously when $\lambda_j^2 - 4\lambda_i\lambda_j \leq 2S$ for any i and j . □

The following estimates are applicable for higher dimension.

Lemma 3.2. *If $n \geq 6$ and $n \leq S \leq \frac{16}{15}n$, then*

$$3(\mathcal{A} - 2\mathcal{B}) \leq (S+4)|\nabla B|^2 + C_3(n)|\nabla B|^2 \mathcal{F}^{\frac{1}{3}}$$

with

$$C_3(n) = \left(\frac{3 - \sqrt{6} - 4p}{\sqrt{6} - 1 + 13p} (6 - \sqrt{6} - 13p)^2 \right)^{\frac{1}{3}}, \quad p = \frac{1}{13(n-2)}.$$

Proof. For any distinct $i, j, k \in \{1, \dots, n\}$, we define

$$\begin{aligned}\phi &= \lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_j\lambda_k - 2\lambda_i\lambda_k, \\ \psi &= \lambda_j^2 - 4\lambda_i\lambda_j.\end{aligned}$$

Firstly, let us estimate ϕ . Without loss of generality, we suppose

$$\lambda_i\lambda_j \leq \lambda_j\lambda_k \leq 0, \quad \lambda_i\lambda_k \geq 0.$$

Define

$$\lambda_i = -x\lambda_j, \lambda_k = -y\lambda_j, \quad x \geq y \geq 0.$$

Now,

$$(3.8) \quad \phi = \lambda_i^2 + \lambda_j^2 + \lambda_k^2 + 2(x + y - xy)\lambda_j^2 \leq S + 4 + 2(x\lambda_j^2 - 1 + (1 - x)y\lambda_j^2 - 1).$$

Let

$$a = x\lambda_j^2 - 1, \quad b = y\lambda_j^2 - 1, \quad c = (1 - x)y\lambda_j^2 - 1,$$

then (3.8) becomes

$$(3.9) \quad \phi \leq S + 4 + 2(a + c).$$

Noting $S \leq \frac{16}{15}n$ and $S \geq \lambda_j^2 + \frac{1}{n-1}(\sum_{k \neq j} \lambda_k)^2$, we deduce

$$(3.10) \quad \lambda_j^2 \leq \frac{16}{15}(n - 1).$$

In the case of $c = (1 - x)y\lambda_j^2 - 1 \geq 0$, which implies $x \leq 1$ and $a, b \geq 0$. By Cauchy inequality and (3.10),

$$\begin{aligned}c &\leq \left(x(1 - x) - \frac{15}{16(n - 1)}\right) \lambda_j^2 \leq \frac{4n - 19}{32n - 17}(x + 1)^2 \lambda_j^2 \leq \left(1 - \frac{16}{5n}\right) \frac{2n - 2}{16n - 1}(x + 1)^2 \lambda_j^2, \\ a &\leq \left(x - \frac{15}{16(n - 1)}\right) \lambda_j^2 \leq \frac{4n - 4}{16n - 1}(x + 1)^2 \lambda_j^2, \\ b &\leq \left(y - \frac{15}{16(n - 1)}\right) \lambda_j^2 \leq \frac{4n - 4}{16n - 1}(y + 1)^2 \lambda_j^2.\end{aligned}$$

For some $\epsilon > 0$ to be defined later,

$$\begin{aligned}(a + c)^3 &= a^3 + c^3 + 3(a^2c + ac^2) \leq a^3 + c^3 + 3\left(a^2c + \frac{\epsilon}{2}a^2c + \frac{1}{2\epsilon}c^3\right) \\ &\leq a^3 + b^3 + 3\frac{2n - 2}{16n - 1} \left[\left(1 + \frac{\epsilon}{2}\right) a^2 \left(1 - \frac{16}{5n}\right) (x + 1)^2 + \frac{1}{\epsilon} b^2 (1 + y)^2\right] \lambda_j^2.\end{aligned}$$

By the definition of \mathcal{F} , we have

$$\begin{aligned}\mathcal{F} &\geq 2(\lambda_i - \lambda_j)^2(\lambda_i\lambda_j + 1)^2 + 2(\lambda_j - \lambda_k)^2(\lambda_j\lambda_k + 1)^2 \\ &= 2(x + 1)^2 \lambda_j^2 a^2 + 2(y + 1)^2 \lambda_j^2 b^2.\end{aligned}$$

Let $\epsilon = \sqrt{\frac{15n-16}{5n-16}} - 1$, then

$$\begin{aligned}
 (3.11) \quad (a+c)^3 &\leq a^3 + |b|^3 + 3 \left(\sqrt{\frac{15n-16}{5n-16}} - 1 \right)^{-1} \frac{2n-2}{16n-1} (a^2(x+1)^2 + b^2(y+1)^2) \lambda_j^2 \\
 &\leq \left(2 + 3 \left(\sqrt{\frac{15n-16}{5n-16}} - 1 \right)^{-1} \right) \frac{2n-2}{16n-1} (a^2(x+1)^2 + b^2(y+1)^2) \lambda_j^2 \\
 &\leq \left(2 + 3 \left(\sqrt{\frac{15n-16}{5n-16}} - 1 \right)^{-1} \right) \frac{n-1}{16n-1} \mathcal{F}.
 \end{aligned}$$

If $c \leq 0$ and $a \geq 0$, then (3.11) holds clearly. Combining (3.9) and (3.11), we have the following estimate

$$(3.12) \quad \phi \leq S + 4 + \left(\left(2 + 3 \left(\sqrt{\frac{15n-16}{5n-16}} - 1 \right)^{-1} \right) \frac{8(n-1)}{16n-1} \mathcal{F} \right)^{\frac{1}{3}}.$$

If $a \leq 0$, then $c \leq 0$ and the above inequality holds clearly. Hence (3.11) holds which is independent of the sign of a, b, c .

Secondly, let's estimate $\psi = \lambda_j^2 - 4\lambda_i\lambda_j$. In the case of $\psi - S - 4 > 0$, then there is a $t > 0$ such that $\lambda_i = -t\lambda_j$. Since

$$S \geq \lambda_i^2 + \lambda_j^2 + \frac{1}{n-2} \left(\sum_{k \neq i, j} \lambda_k \right)^2 = \frac{n-1}{n-2} \lambda_i^2 + \frac{n-1}{n-2} \lambda_j^2 + \frac{2}{n-2} \lambda_i \lambda_j,$$

then

$$\begin{aligned}
 (3.13) \quad \psi &\leq S - 4\lambda_i\lambda_j - \frac{n-1}{n-2} \lambda_i^2 - \frac{2}{n-2} \lambda_i\lambda_j - \frac{1}{n-2} \lambda_j^2 \\
 &= S + \left(-\frac{n-1}{n-2} t^2 + \frac{4n-6}{n-2} t - \frac{1}{n-2} \right) \lambda_j^2.
 \end{aligned}$$

Since $n \geq 6$ and (3.10), we have

$$\begin{aligned}
 (3.14) \quad \psi &\leq S + 4 + \left(-\frac{n-1}{n-2} t^2 + \frac{4n-6}{n-2} t - \frac{1}{n-2} \right) \lambda_j^2 - \frac{15}{4(n-1)} \lambda_j^2 \\
 &\leq S + 4 + \left(-\frac{n-1}{n-2} t^2 + \frac{4n-6}{n-2} t - \frac{4}{n-2} \right) \lambda_j^2.
 \end{aligned}$$

By Cauchy inequality,

$$-\frac{n-1}{n-2} t^2 + \frac{4n-6}{n-2} t - \frac{4}{n-2} \leq \left(t - \frac{12}{13(n-2)} \right) (4-t).$$

By (3.13),

$$\psi \leq S - 4\lambda_i\lambda_j - \lambda_i^2 = S + (4t - t^2) \lambda_j^2,$$

combining (3.14), we have

$$\begin{aligned}
 (\psi - S - 4)^3 &\leq ((4t - t^2)\lambda_j^2 - 4)^2 \left(-\frac{n-1}{n-2}t^2 + \frac{4n-6}{n-2}t - \frac{4}{n-2} \right) \lambda_j^2 \\
 (3.15) \quad &\leq ((4t - t^2)\lambda_j^2 - (4-t))^2 \left(t - \frac{12}{13(n-2)} \right) (4-t)\lambda_j^2 \\
 &= \left(t - \frac{12}{13(n-2)} \right) (4-t)^3 (t\lambda_j^2 - 1)^2 \lambda_j^2.
 \end{aligned}$$

Now we define an auxiliary function

$$\omega(t, \xi) = \left(t - \frac{12}{13(n-2)} \right) (4-t)^3 - \xi(1+t)^2.$$

Then there exists the smallest ξ such that

$$\sup_t \omega(t, \xi) = 0.$$

For any t_0 satisfying $\partial_t \omega(t_0, \xi) = \omega(t_0, \xi) = 0$, we solve the equations to get

$$t_0 = \sqrt{6 + 54p + 9p^2} - 2 + 3p,$$

$$\xi = \frac{1}{1+t_0} \left(2 - 2t_0 + \frac{18}{13(n-2)} \right) (4-t_0)^2,$$

here $p = \frac{1}{13(n-2)}$. Since

$$t_0 \geq \sqrt{6} + 10p - 2 + 3p = \sqrt{6} - 2 + 13p,$$

then

$$\xi \leq 2 \frac{3 - \sqrt{6} - 4p}{\sqrt{6} - 1 + 13p} (6 - \sqrt{6} - 13p)^2,$$

Hence

$$(3.16) \quad \left(t - \frac{12}{13(n-2)} \right) (4-t)^3 \leq 2 \frac{3 - \sqrt{6} - 4p}{\sqrt{6} - 1 + 13p} (6 - \sqrt{6} - 13p)^2 (1+t)^2.$$

Noting

$$\mathcal{F} \geq 2(\lambda_i - \lambda_j)^2 (1 + \lambda_i \lambda_j)^2 = 2(t+1)^2 \lambda_j^2 (t\lambda_j^2 - 1)^2$$

and (3.15), (3.16), we have

$$(3.17) \quad (\psi - S - 4)^3 \leq \frac{3 - \sqrt{6} - 4p}{\sqrt{6} - 1 + 13p} (6 - \sqrt{6} - 13p)^2 \mathcal{F}.$$

If $\psi - S - 4 \leq 0$, the above inequality holds clearly. Let

$$C_3(n) = \left(\frac{3 - \sqrt{6} - 4p}{\sqrt{6} - 1 + 13p} (6 - \sqrt{6} - 13p)^2 \right)^{\frac{1}{3}}.$$

By a calculation $C_3(n)^3 \geq \left(2 + 3 \left(\sqrt{\frac{15n-16}{5n-16}} - 1\right)^{-1}\right) \frac{8(n-1)}{16n-1}$ for $n \geq 6$. In fact, both sides of the above inequality are increase in n , we only need to check the case $n = 6$ and

$$C_3(7)^3 \geq \frac{7 + 3\sqrt{3}}{4} = \lim_{n \rightarrow \infty} \left(2 + 3 \left(\sqrt{\frac{15n-16}{5n-16}} - 1\right)^{-1}\right) \frac{8(n-1)}{16n-1}.$$

Combining (3.12) and (3.17), we finally obtain

$$\begin{aligned} 3(\mathcal{A} - 2\mathcal{B}) &\leq \sum_{i,j,k \text{ distinct}} h_{ijk}^2 (\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_j\lambda_k - 2\lambda_i\lambda_k) \\ &\quad + 3 \sum_{j,i \neq j} h_{ij}^2 (\lambda_j^2 - 4\lambda_i\lambda_j) \\ &\leq \sum_{i,j,k \text{ distinct}} h_{ijk}^2 (S + 4 + C_3(n)\mathcal{F}^{1/3}) + 3 \sum_{j,i \neq j} h_{ij}^2 (S + 4 + C_3(n)\mathcal{F}^{1/3}) \\ &\leq (S + 4)|\nabla B|^2 + C_3(n)|\nabla B|^2 \mathcal{F}^{1/3}. \end{aligned}$$

□

4. PROOF OF THEOREMS

4.1. Proof of Theorem 1.1. Since we already have known result for lower dimension, we assume the dimension $n \geq 4$. By (2.1), (2.5) and (2.6), we have

$$\begin{aligned} \int_M \sum_{i,j,k,l} h_{ijkl}^2 &\geq \frac{3}{2} \int_M (Sf_4 - f_3^2 - S^2 - S(S-n)) + \int_M \frac{3S(S-n)^2}{2(n+4)} \\ &= \frac{3}{2} \int_M (Sf_4 - f_3^2 - S^2) - \frac{3}{2} \int_M |\nabla B|^2 + \int_M \frac{3S(S-n)^2}{2(n+4)} \\ &= \frac{3}{2} \int_M (\mathcal{A} - 2\mathcal{B}) + \frac{3}{8} \int_M |\nabla S|^2 - \frac{3}{2} \int_M |\nabla B|^2 + \int_M \frac{3S(S-n)^2}{2(n+4)}. \end{aligned}$$

Combining (2.5), for some fixed $0 < \theta < 1$ to be defined later, we have

$$\begin{aligned} (4.1) \quad &\frac{3\theta}{2} \int_M (\mathcal{A} - 2\mathcal{B}) + \frac{3\theta}{8} \int_M |\nabla S|^2 + \frac{3}{4}(1-\theta) \int_M \mathcal{F} + \int_M \frac{3S(S-n)^2}{2(n+4)} \\ &\leq \frac{3\theta}{2} \int_M |\nabla B|^2 + \int_M |\nabla^2 B|^2. \end{aligned}$$

Together with (2.3), (4.1) and Lemma 3.1, we obtain

$$\begin{aligned}
& \frac{3}{4}(1-\theta) \int_M \mathcal{F} + \int_M \frac{3S(S-n)^2}{2(n+4)} - \left(\frac{3}{2} - \frac{3\theta}{8}\right) \int_M |\nabla S|^2 \\
& \leq \int_M \left(S - 2n - 3 + \frac{3\theta}{2}\right) |\nabla B|^2 + \left(3 - \frac{3\theta}{2}\right) \int_M (\mathcal{A} - 2\mathcal{B}) \\
(4.2) \quad & \leq \int_M \left(S - 2n - 3 + \frac{3\theta}{2}\right) |\nabla B|^2 + \left(1 - \frac{\theta}{2}\right) \int_M (2S|\nabla B|^2 + C_1|\nabla B|^2 f^{\frac{1}{3}}) \\
& \leq \int_M \left((3-\theta)S - 2n - 3 + \frac{3\theta}{2}\right) |\nabla B|^2 + \frac{3}{4}(1-\theta) \int_M \mathcal{F} \\
& \quad + \frac{4}{9}C_1^{\frac{3}{2}} \left(1 - \frac{\theta}{2}\right)^{\frac{3}{2}} (1-\theta)^{-\frac{1}{2}} \int_M |\nabla B|^3,
\end{aligned}$$

where we have used Young's inequality in the last step of (4.2), then

$$\begin{aligned}
(4.3) \quad \int_M \frac{3S(S-n)^2}{2(n+4)} & \leq \int_M \left((3-\theta)S - 2n - 3 + \frac{3\theta}{2}\right) |\nabla B|^2 \\
& \quad + \left(\frac{3}{2} - \frac{3\theta}{8}\right) \int_M |\nabla S|^2 + C_2(n, \theta) \int_M |\nabla B|^3,
\end{aligned}$$

where $C_2(n, \theta) = \frac{4}{9}C_1^{\frac{3}{2}}(1 - \frac{\theta}{2})^{\frac{3}{2}}(1 - \theta)^{-\frac{1}{2}}$.

By (2.1), for some $\epsilon > 0$ to be defined later, we have

$$\begin{aligned}
(4.4) \quad \int_M |\nabla B|^3 & = \int_M S(S-n)|\nabla B| + \frac{1}{2} \int_M |\nabla B| \Delta S \\
& = \int_M S(S-n)|\nabla B| - \frac{1}{2} \int_M \nabla |\nabla B| \cdot \nabla S \\
& \leq \int_M S(S-n)|\nabla B| + \epsilon \int_M |\nabla^2 B|^2 + \frac{1}{16\epsilon} \int_M |\nabla S|^2.
\end{aligned}$$

Combining (2.3) and (2.4), we obtain

$$\int_M |\nabla^2 B|^2 \leq \int_M ((\alpha+1)S - 2n - 3)|\nabla B|^2 + \frac{3}{2} \int_M |\nabla S|^2.$$

With the help of the above inequality, (4.4) becomes

$$\begin{aligned}
(4.5) \quad \int_M |\nabla B|^3 & \leq \int_M S(S-n)|\nabla B| \\
& \quad + \int_M \epsilon((\alpha+1)S - 2n - 3)|\nabla B|^2 + \left(\frac{3\epsilon}{2} + \frac{1}{16\epsilon}\right) \int_M |\nabla S|^2.
\end{aligned}$$

Multiplying S on the both sides of (2.1), and integrating by parts, we see

$$\begin{aligned}
 (4.6) \quad \frac{1}{2} \int_M |\nabla S|^2 &= \int_M S^2(S-n) - \int_M S|\nabla B|^2 \\
 &= \int_M S(S-n)^2 + n \int_M S(S-n) - \int_M S|\nabla B|^2 \\
 &= \int_M (n-S)|\nabla B|^2 + \int_M S(S-n)^2.
 \end{aligned}$$

Combining (4.3), (4.5) and (4.6), we get

$$\begin{aligned}
 (4.7) \quad 0 &\leq \int_M \left((3-\theta)S - 2n - 3 + \frac{3\theta}{2} + C_2\epsilon((\alpha+1)S - 2n - 3) \right) |\nabla B|^2 \\
 &\quad + C_2 \int_M S(S-n)|\nabla B| + \left(\frac{3}{2} - \frac{3\theta}{8} + C_2 \left(\frac{3\epsilon}{2} + \frac{1}{16\epsilon} \right) \right) \int_M |\nabla S|^2 \\
 &\quad - \int_M \frac{3S(S-n)^2}{2(n+4)} \\
 &\leq \int_M \left((3-\theta)S - 2n - 3 + \frac{3\theta}{2} + C_2\epsilon((\alpha+1)S - 2n - 3) \right) |\nabla B|^2 \\
 &\quad + C_2 \int_M S(S-n)|\nabla B| - \int_M (S-n) \left(3 - \frac{3\theta}{4} + C_2(3\epsilon + \frac{1}{8\epsilon}) \right) |\nabla B|^2 \\
 &\quad + \left(3 - \frac{3\theta}{4} + C_2(3\epsilon + \frac{1}{8\epsilon}) - \frac{3}{2(n+4)} \right) \int_M S(S-n)^2 \\
 &= \int_M \left((1-\theta)n - 3 + \frac{3\theta}{2} + C_2\epsilon(\alpha n - n - 3) \right. \\
 &\quad \left. - (S-n) \left(\frac{\theta}{4} + C_2\epsilon(2-\alpha) + \frac{C_2}{8\epsilon} \right) \right) |\nabla B|^2 \\
 &\quad + \left(3 - \frac{3\theta}{4} + C_2 \left(3\epsilon + \frac{1}{8\epsilon} \right) - \frac{3}{2(n+4)} \right) \int_M S(S-n)^2 \\
 &\quad + C_2 \int_M S(S-n)|\nabla B|.
 \end{aligned}$$

By the assumption $n \leq S \leq n + \delta(n)$, Cauchy-Schwartz inequality and (2.1), we have

$$\begin{aligned}
 (4.8) \quad \int_M S(S-n)|\nabla B| &\leq 2(n+\delta)\epsilon \int_M S(S-n) + \frac{1}{8(n+\delta)\epsilon} \int_M S(S-n)|\nabla B|^2 \\
 &= \int_M \left(2(n+\delta)\epsilon + \frac{S(S-n)}{8(n+\delta)\epsilon} \right) |\nabla B|^2 \leq \int_M \left(2(n+\delta)\epsilon + \frac{S-n}{8\epsilon} \right) |\nabla B|^2.
 \end{aligned}$$

From (2.1), (4.7) and (4.8) we see that

$$(4.9) \quad 0 \leq \int_M \left((1-\theta)n - 3 + \frac{3\theta}{2} + O(\epsilon) \right) |\nabla B|^2,$$

where we choose $\delta = \varepsilon^2$. We could choose θ close to 1, then it is easily seen that there exists $\varepsilon > 0$, such that the coefficient of the integral in (4.9) is negative. This forces $|\nabla B| = 0$. We now complete the proof of Theorem 1.1.

4.2. Proof of Theorem 1.2. We assume $n \geq 6$. In the proof of Theorem 1.1, replacing Lemma 3.1 by Lemma 3.2 in (4.2), we have

$$\begin{aligned}
 & \frac{3}{4}(1-\theta) \int_M \mathcal{F} + \int_M \frac{3S(S-n)^2}{2(n+4)} - \left(\frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 \\
 & \leq \int_M (S - 2n - 3 + \frac{3\theta}{2}) |\nabla B|^2 + (1 - \frac{\theta}{2}) \int_M ((S+4) |\nabla B|^2 + C_3 |\nabla B|^2 \mathcal{F}^{\frac{1}{3}}) \\
 (4.10) \quad & \leq \int_M \left((2 - \frac{\theta}{2})S - 2n + 1 - \frac{\theta}{2} \right) |\nabla B|^2 + \frac{3}{4}(1-\theta) \int_M \mathcal{F} \\
 & \quad + \frac{4}{9} C_3^{\frac{3}{2}} \left(1 - \frac{\theta}{2} \right)^{\frac{3}{2}} (1-\theta)^{-\frac{1}{2}} \int_M |\nabla B|^3.
 \end{aligned}$$

Combining (4.5) and (4.6) we see

$$\begin{aligned}
 (4.11) \quad & 0 \leq \int_M \left[-\frac{\theta}{2}(n+1) + 1 + C_4 \epsilon (\alpha n - n - 3) \right. \\
 & \quad \left. - (S-n) \left(1 - \frac{\theta}{4} + C_4 \epsilon (2-\alpha) + \frac{C_4}{8\epsilon} \right) \right] |\nabla B|^2 \\
 & \quad + \left(3 - \frac{3\theta}{4} + C_4 \left(3\epsilon + \frac{1}{8\epsilon} \right) - \frac{3}{2(n+4)} \right) \int_M S(S-n)^2 + C_4 \int_M S(S-n) |\nabla B|,
 \end{aligned}$$

where $C_4 = C_4(n, \theta) = \frac{4}{9} C_3^{\frac{3}{2}} \left(1 - \frac{\theta}{2} \right)^{\frac{3}{2}} (1-\theta)^{-\frac{1}{2}}$.

Assume $n \leq S \leq n + \delta(n)$, by (4.8), we have

$$\begin{aligned}
(4.12) \quad 0 &\leq \int_M \left[-\frac{\theta}{2}(n+1) + 1 + C_4\epsilon(\alpha n + n - 3 + 2\delta) \right. \\
&\quad \left. - (S-n)\left(1 - \frac{\theta}{4} + C_4\epsilon(2-\alpha)\right) \right] |\nabla B|^2 \\
&\quad + \left(3 - \frac{3\theta}{4} + C_4\left(3\epsilon + \frac{1}{8\epsilon}\right) - \frac{3}{2(n+4)} \right) \int_M S(S-n)^2 \\
&\leq \int_M \left[-\frac{\theta}{2}(n+1) + 1 + C_4\epsilon(\alpha n + n - 3 + 2\delta) \right. \\
&\quad \left. - (S-n)\left(1 - \frac{\theta}{4} + C_4\epsilon(2-\alpha)\right) \right] |\nabla B|^2 \\
&\quad + \left(3 - \frac{3\theta}{4} + C_4\left(3\epsilon + \frac{1}{8\epsilon}\right) - \frac{3}{2(n+4)} \right) \delta \int_M |\nabla B|^2 \\
&= \left(-\frac{\theta}{2}(n+1) + 1 + C_4\epsilon(\alpha n + n - 3 + 5\delta) + \frac{C_4}{8\epsilon}\delta \right. \\
&\quad \left. + \left(\frac{3(2n+5)}{2(n+4)} - \frac{3\theta}{4} \right) \delta \right) \int_M |\nabla B|^2 \\
&\quad - \int_M \left(1 - \frac{\theta}{4} + C_4\epsilon(2-\alpha) \right) (S-n) |\nabla B|^2.
\end{aligned}$$

Let $\epsilon = \sqrt{\frac{\delta}{8(\alpha n + n - 3 + 5\delta)}}$ and $\theta = 0.84$, then

$$C_4(n) = \frac{4}{9} \times 0.58^{3/2} \times 0.16^{-1/2} \times \sqrt{\frac{3 - \sqrt{6} - 4p}{\sqrt{6} - 1 + 13p}} (6 - \sqrt{6} - 13p),$$

where $p = \frac{1}{13(n-2)}$. We have $C_4(n) \leq \lim_{l \rightarrow \infty} C_4(l) \leq 1.1$. Combining $\delta(n) \leq \frac{n}{15}$ and $\alpha = \frac{\sqrt{17}+1}{2}$ we obtain $0.79 + C_4\epsilon(2-\alpha) \geq 0$. From (4.12) we get

$$(4.13) \quad 0 \leq \left(-0.42n + 0.58 + C_4\sqrt{\frac{\delta}{2}(\alpha n + n - 3 + 5\delta)} + \left(\frac{3(2n+5)}{2(n+4)} - 0.63 \right) \delta \right) \int_M |\nabla B|^2.$$

If $\delta(n) = \frac{n}{23}$, then the coefficient of the integral in (4.13) is negative, hence, $|\nabla B| \equiv 0$, $S \equiv n$. The proof is complete.

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INSTITUTE OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI 200433, CHINA

E-mail address: 09110180013@fudan.edu.cn

E-mail address: ylxin@fudan.edu.cn